MATH 2050C Lecture 22 (Apr 12)

Final Exam: May 5, 2022 (Thur.) 12:30 - 2:30PM

An email will be sent about the detailed arrangements.

"Uniform" Continuity

Recall: Let $f: A \to \mathbb{R}$.

① f is cts at $c \in A$

implicitly depend on f and c

$\overset{\text{def}^{\text{n}}}{\Longleftrightarrow}$ $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t.

$$|f(x) - f(c)| < \varepsilon \quad \forall |x - c| < \delta, x \in A$$

② f is cts on A

$\overset{\text{def}^{\text{n}}}{\Longleftrightarrow}$ f is cts at EVERY $c \in A$

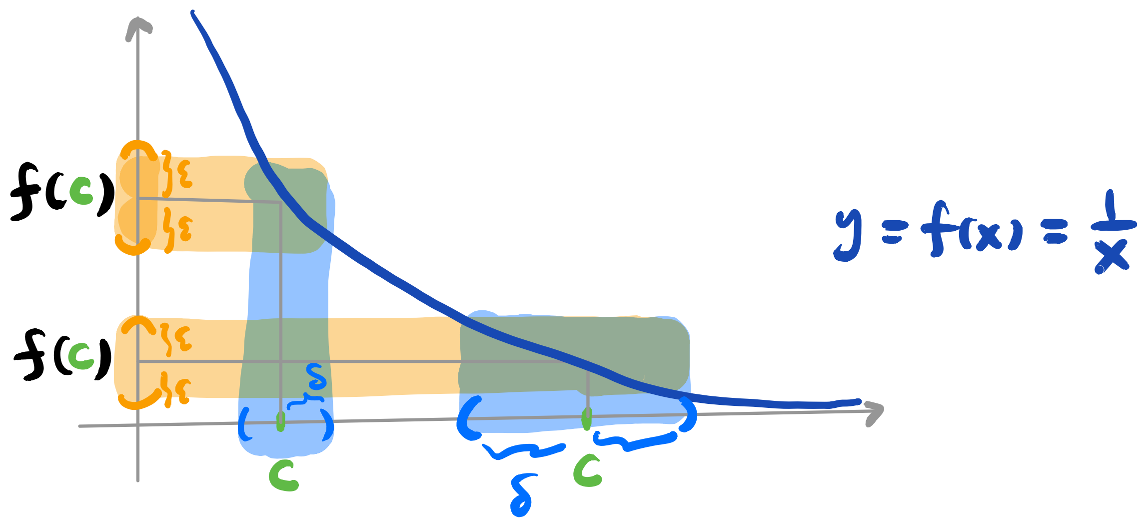
$\Longleftrightarrow$ $\forall c \in A, \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon, c) > 0$ s.t.

$$|f(x) - f(c)| < \varepsilon \quad \forall |x - c| < \delta, x \in A$$

Remark: Generally speaking, the choice of δ depends on both ε and c.

Example 1 : $f : (0,\infty) \to \mathbb{R}$

$f(x) := \frac{1}{x}$ is cts on $(0,\infty)$



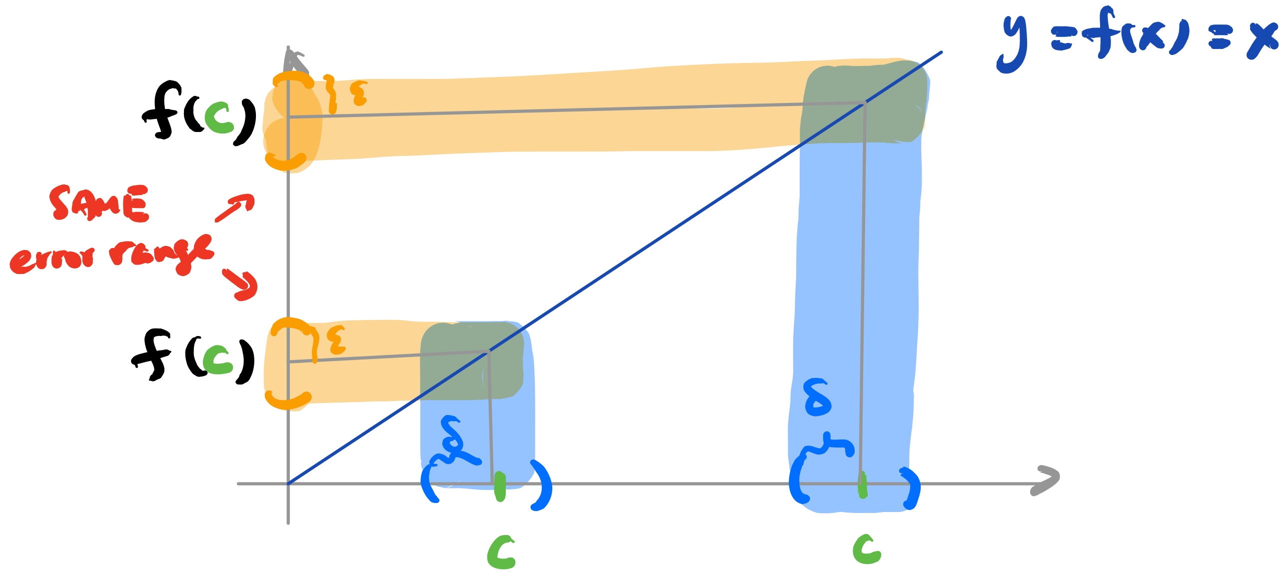
For the SAME $\varepsilon > 0$, when $c \approx 0$, then we need a smaller $\delta > 0$ around c to achieve the desire error range (specified by ε).

$\leadsto$ NOT uniformly continuous

$\because$ δ is not "uniform" in c.

Example 2: $f:(0,\infty)\to\mathbb{R}$

$f(x) := x$ cts on $(0,\infty)$



For the SAME $\varepsilon>0$, we can choose ONE $\delta>0$ s.t. it works for all the points c.

ie the choice of δ does not depend on c (but still depends on ε)

$\rightsquigarrow$ uniformly continuous

Def$^{\underline{n}}$: $f : A \to \mathbb{R}$ is uniformly continuous (on A)

does NOT depend on u, v

$\iff \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t.

$|f(u) - f(v)| < \varepsilon$ whenever $u, v \in A$ and $|u - v| < \delta$

Remark: 1) Fix $v = c \in A$. then clearly

uniformly cts on A $\Rightarrow$ cts on A

$\Leftarrow\!\!\!\!\!\times$ c.f. Example 1

2) Uniform continuity is a "global" concept, i.e. it does NOT make sense to talk about uniform continuity at one point.

Q: How to decide whether $f : A \to \mathbb{R}$ is uniformly cts?

Of course, if f is not cts everywhere on A, then f CANNOT be uniformly cts.

Prop: $f: A \to \mathbb{R}$ is NOT uniformly cts

$\iff \exists\, \varepsilon_0 > 0$ s.t. $\forall\, \delta > 0$, $\exists\, u_\delta, v_\delta \in A$

s.t. $|u_\delta - v_\delta| < \delta$ BUT $|f(u_\delta) - f(v_\delta)| \geq \varepsilon_0$

$\iff \exists\, \varepsilon_0 > 0$ and seq $(u_n), (v_n)$ in A

s.t. $|u_n - v_n| < \frac{1}{n}$ BUT $|f(u_n) - f(v_n)| \geq \varepsilon_0$

$\forall\, n \in \mathbb{N}$

Proof: Taking negation of def^n & choose $\delta = \frac{1}{n}$.

$\square$

Remark: This Proposition is useful in proving a function $f: A \to \mathbb{R}$ is NOT uniformly cts.

Example 1 (again): The function $f(x) = \frac{1}{x}$ is NOT uniformly cts on $(0, \infty)$.

Proof: Take $\varepsilon_0 = \frac{1}{2}$ and $u_n := \frac{1}{n}$, $v_n := \frac{1}{n+1}$.

Note $|u_n - v_n| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \frac{1}{n}$

BUT $|f(u_n) - f(v_n)| = |n - (n+1)| = 1 > \varepsilon_0 = \frac{1}{2}$

By Prop. we are done.

$\square$

Exercise: Show that $f(x) = \frac{1}{x}$ is uniformly cts on $[a, \infty)$ for any fixed $a > 0$.

Q: In terms of uniform continuity, can we say more when $f: A \to \mathbb{R}$ is defined on an interval A?

A: Yes, we will talk about two results

$\left\{ \begin{array}{l} \text{Uniform Continuity Thm} \\ \text{Continuous Extension Thm} \end{array} \right.$

Recall: uniformly cts on A $\Rightarrow$ cts on A

$\not\Leftarrow$ c.f. Example 1

Uniform Continuity Thm

closed & bdd interval

If $f: [a,b] \to \mathbb{R}$ is cts on $[a,b]$, then f is uniformly cts on $[a,b]$.

Proof: We shall argue by contradiction.

Suppose f is NOT uniformly cts on $[a,b]$.

By previous Prop., then

$\exists$ $\varepsilon_0 > 0$ and seq. (u_n), (v_n) in $[a,b]$

s.t. $|u_n - v_n| < \frac{1}{n}$ BUT $|f(u_n) - f(v_n)| \geq \varepsilon_0$

$\forall n \in \mathbb{N}$

By Bolzano-Weierstrass Thm, $\exists$ subseq.

(u_{n_k}) of (u_n) s.t.

$$\lim_{k\to\infty} (u_{n_k}) = u^* \in [a,b]$$

Note that $\forall k \in \mathbb{N}$, we have

$$|u_{n_k} - v_{n_k}| < \frac{1}{n_k} \overset{k\to\infty}{\Longrightarrow} \lim_{k\to\infty} (v_{n_k}) = u^*$$

Now, by construction, we have $\forall k \in \mathbb{N}$

$$|f(u_{n_k}) - f(v_{n_k})| \geq \varepsilon_0 > 0$$

Taking $k\to\infty$ and using the continuity of f (at u^*)

$$\Rightarrow \underbrace{|f(u^*) - f(u^*)|}_{=0} \geq \varepsilon_0 > 0$$

Contradiction arises!

$\square$

Q: Given a cts $f : (a,b) \to \mathbb{R}$, when can we extend it continuously to a function

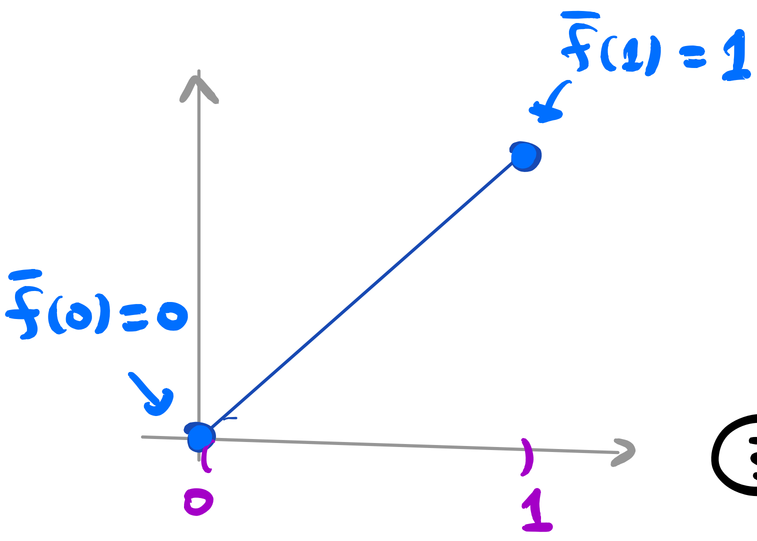
$$\bar{f} : [a,b] \to \mathbb{R} \ ?$$

i.e. $\bar{f}(x) = f(x), \quad x \in (a,b)$.

Pictures

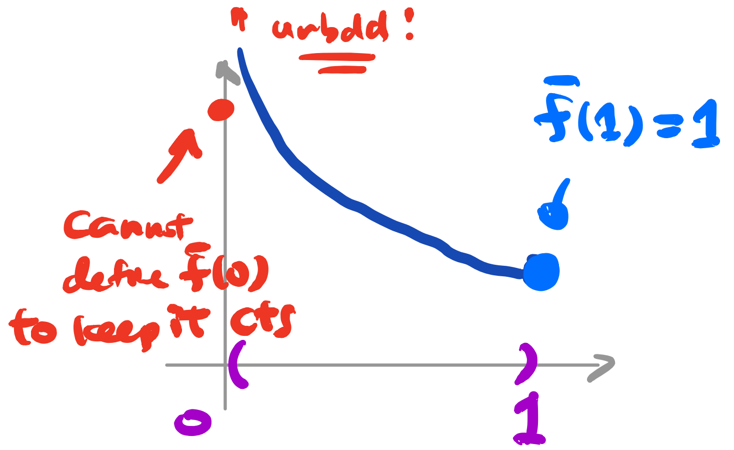
① $f(x) = x, \quad x \in (0,1)$

$\rightsquigarrow \bar{f}(x) = x, \quad x \in [0,1]$



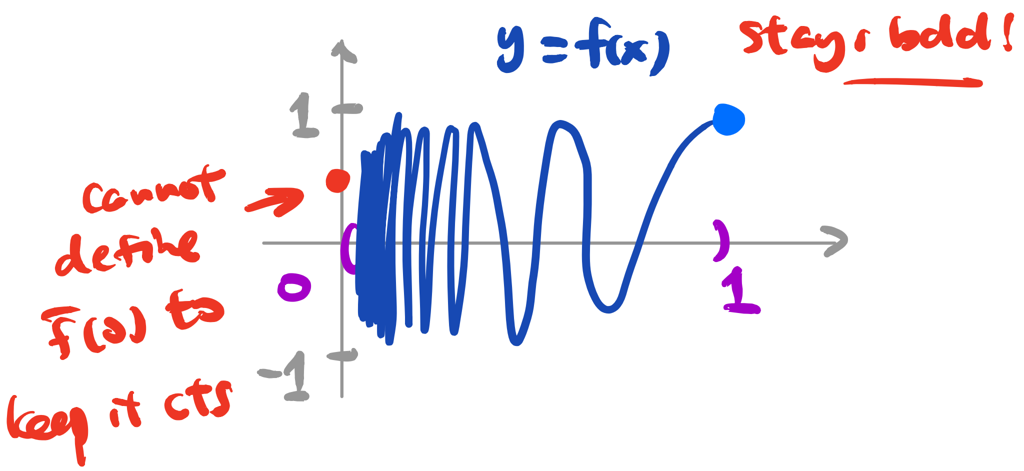
Yes!

② $f(x) = \frac{1}{x}, \quad x \in (0,1)$



No!

③ $f(x) = \sin\frac{1}{x}, \quad x \in (0,1)$



Continuous Extension Thm

Let $f : (a,b) \to \mathbb{R}$ be a cts function.

If f is uniformly cts on (a,b), $\cdots (*)$

then $\exists$ an "extension" $\bar{f} : [a,b] \to \mathbb{R}$

s.t (i) $\bar{f}(x) = f(x) \quad \forall x \in (a,b)$

(ii) $\bar{f}$ is cts on $[a,b]$

Remark: (1) $\bar{f}$ is uniformly cts on $[a,b]$

by Uniform Continuity Thm.

($\Rightarrow$ $(*)$ is necessary)

(2) Such an extension $\bar{f}$ is unique.

(we will see why in the proof.)

We will the following result in the proof of

Continuous Extension Thm.

Prop: Let $f: A \to \mathbb{R}$ be a uniformly cts function. Suppose (x_n) is a Cauchy seq. in A. THEN. $(f(x_n))$ must also be a Cauchy seq.

In other words, Cauchy seq. are "preserved" by uniformly cts functions.

Proof: Let $\varepsilon > 0$. By def.$^{\underline{n}}$ of unif. continuity, $\exists\ \delta = \delta(\varepsilon) > 0$ s.t.

(#) ... $|f(u) - f(v)| < \varepsilon$, $\forall\ u, v \in A$, $|u - v| < \delta$

Suppose (x_n) is a Cauchy seq in A.

By def.$^{\underline{n}}$ of Cauchy seq., for the $\delta > 0$ above, $\exists\ H = H(\delta) \in \mathbb{N}$ s.t.

$$|x_m - x_n| < \delta \qquad \forall\ m, n \geq H$$

By (#), $|f(x_m) - f(x_n)| < \varepsilon \quad \forall m, n \geq H$

i.e. $(f(x_n))$ is Cauchy.

$\square$